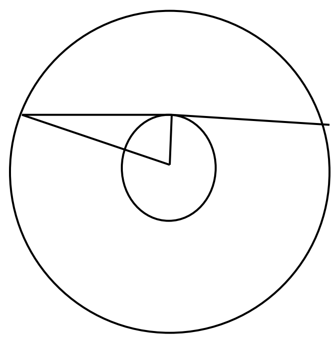
2013 Junior High Mathematics Contest

Solutions:

7^{th} Grade Individual Contest:

1) Solve for x. The roots are -1/3 and 5/2. (-1/3)*(5/2) = **-5/6**
2) The difference between each even and odd term is 1. Since there are 1000 of them, our answer is just **1000.**
3) Determine the prime factorization of 720: It is 2^4*3^2*5^1.
   Thus, the sum of the factors is (2^0 + 2^1 + 2^2 + 2^3 + 2^4)(3^0+3^1+3^2)(5^0 + 5^1) = **2418.**
4) Ignore the tens and hundreds place in 148. This problem is equivalent to asking for the last digit of 8^65342. You know that the unit digits of the power of 8 cycle: 8, 4, 2, 6. Since 65342 is 2 mod 4, our answer is **4.**
5) 

Here, we have a right angle, in which the hypotenuse is the radius of the outer circle (26), while the shorter leg is the radius of the smaller circle (10). Therefore, half the common chord has length 24. The chord's length is 24*2 = **48.**

6) Saieesh can write ½ a math test in a hour, Kevin can write 1/3 a math test in a hour, and Jose can write 1/12 a math test in a hour. Together, they can write (1/2) + (1/3) + (1/12) of a math test in a hour, which is 11/12. So, it takes them 1/(11/12) = **12/11** hour.
7) If person A has m donuts and person B has n donuts, then the maximum amount that they cannot make is defined to be mn-(m+n). This is known as the Chicken McNugget theorem. So, our answer is just 8*9 – (8+9) = 72-17 = **55.**
8) The circumradius of a right triangle is half of its hypotenuse which is 5. The inradius can be calculated by using the formula A = rs, where A is the area of the triangle, r is the inradius and s is the semi perimeter of the triangle. So, 24 = r*12. r = 2. Thus, our answer is **2.5.**
9) Simple rearrangement question: 5! / 2! = **60.**
10) Principles of Inclusion/Exclusion: 135+ 101+ 34 – (47+20+10) + x= 200.
    X = 7.
11) 4!/4 give you 3! = 6. But, divide by 2 because you can flip the keychain. Answer is **3.**
12) Suppose the original numbers are 3x and 4x. Then, (3x+4) / (4x+4) = 4/5. X = 4. Thus, our original numbers are 12 and 16. Our answer is **16.**
13) If the average of the bases is 6, then the sum is 12. Since the perimeter is 15, the length of each leg is just (15-12)/2 = **3/2**
14) Think about it. Answer is **121.**

15) Since a-b = 0, a=b. Since ab = 2, a = sqrt2 and b = sqrt2. (a/b) + (b/a) + 2a^2 + b^2 = 2 + 2 +2 + 2 = **8.**
16) This is done by dividing 500 by powers of 5 as long as the value is greater than 1. So, (500/5)+(500/25)+(500/125)=100+20+4=**124**
17) The lengths of the right the right triangle would be 5,12,13 and 13,84,85.
    The sum of the perimeters is 30 + 182 = **212**
18) The area formed is a trapezoid. The vertical height is 3, and the two bases have lengths of 2 and 14. So, the area is 3(2+14)/2=24
19) The numbers 3,4,……48 each appear 3 times. Their sum is 3(3+4+….+48). The numbers 2 and 49 appear each 2 times. Finally, 1 and 50 each appear once.
    So, our answer is 3(3+4…+48) + 2(2+49) + 1(1+50) = **3672.**
20) The prime factorization of 18,000 is 2^4*3^2 * 5^3. The number of perfect squares is :
    (2^2)^2 * (3^2)^1 * (5^2)^1.
    The total number of perfect square factors would be (2+1)(1+1)(1+1) = 3*2*2 = **18.**

2013 Junior High Mathematics Contest

Solutions:

8^{th} Grade Individual Contest

1) Let the two numbers be x and y. Then, x^2 + y^2 = 42, while (xy)^2 = 120 or xy = sqrt(120). Hence, (x+y)^2 = 42 + 2(sqrt120) 42 + 2(2sqrt30) = **42 + 4sqrt30**
2) Let the daughter be age x, her father is 5x. So, (5x)^2 + (x)^2 = 2106. X = daughter's age = **9**
3) Christine can finish 1/10 of a window in a hour, while Angela can finish 1/5 of a window in a hour. Together, they finish 3/10 of a window in a hour. They can finish a window in **10/3** min.
4) The number of squares would be 1^2 + 2^2 + 3^2 + 4^2 + … + 8^2 = (8)(9)(17)/6 = **204.**
5) The maximum possible of area would be when the rectangle has sides of length 31 and 29. The area of this rectangle is 31*29 = **899.**
6) Let the hexagon have a side x. Let the square have side length y. So, 6x = 4y. x = 2y/3. The ratio of the hexagon's area to the square's area would be (2/3)^2 = **4/9.**
7) This would just be the total number of combinations subtracted by the number of combinations in which we have the ham and tomatoes together. 4*3*2 – (1*1*2) = **22.**
8) The dimensions of the largest possible length is 28,10,10. The answer is **28.**
9) Suppose the number is 10x+ y. Then, the reverse of the number is 10y+x. So, we can rewrite the problem to be 10x+y + 27 = 10y+x.
   9y-9x = 27 -→ y-x = 3.
   The possible numbers are 14, 25, 36, 47, 58, 69.
   Our answer is 14+25+36+47+58+69 = 83+83+83 = **249**
10) Using Fermat's Little Theorem, we can break this up into several mods.
    2^{20} mod 9 = 4
    3^{30} mod 9 = 0
    4^{40} mod 9 = 4
    5^{50} mod 9 = 7
    6^{60} mod 9 = 0
    Adding this together, 4+4+7=15, 15 mod 9 = 6
11) 1 – Probability that you don't flip a tail = 1 – (1/32) = **31/32**
12) We can take the sum of the floors of each of these fractions:
    (500/2) + (500/4) + (500/8) + (500/16) + (500/32) + (500/64) + (500/128) + (500/256) =
    = 250 + 125+ 62 + 31 + 15 + 7 + 3 + 1 = **494.**
13) We can think of this problem as 9 sticks and 2 dividers. There are 11 C 2 = 11*10/2 = **55.**
14) The first pick can be from any suit, so 52/52. The second pick should match the first suit, so 12/51, 11/50, and so on. The probability is 52/52*12/51*11/50*10/49, or 44/4165.
15) **2400**

16) $x^2 + y = 12 = y^2 + x$
x^2 + y = y^2 + x
x^2 – y^2 = x – y
(x+y)(x-y) = x-y
x- y = 0, x+y = 1.
The first case is when x = y.
X^2 + x = 12 -> x^2 + x – 12 = 0.
(x+4)(x-3) = 0. X = -4, 3.
Therefore, two points are (-4,-4) (3,3).
The other case when x + y = 1 does not produce integer solutions, so we can disregard it.
Thus, the answer is **(-4,-4) and (3,3).**
17) The hexagon can be divided into 6 equilateral triangles. The length of the apothem of the hexagon is just the height of one of the equilateral triangles with side length 3. That would be just **3*sqrt3/2**
18) This is just the Chicken McNugget theorem. 8*9 – (8+9) = **55.**
19) We can express our probability as ((5C2)*(5C2)) / (10C4) = 100 / 210 = **10/21.**
20) We can write (1/x) + (1/y) as (x+y)/xy and equate it to 1/7.
Thus, xy = 7x + 7y.
Xy – 7x – 7y = 0
(x-7)(y-7) = 49.

2013 Junior High Mathematics Contest

Solutions: 7th Grade Team Contest

1) The cube root of 8^8 is just 2^8 =**256.**

2) 2+4+6+….+100 = 2(1+2+3+…+50)= **2550.**

3) The worst possible condition is to pick the first 4 marbles such that they are each of a different color. Thus, the last marble must be a repetition of a color that was already picked. Our answer is **5.**

4) 30% of 150 is 45. Thus, we are looking for (45/75)*100 = **60.**

5) A regular hexagon can be divided into 6 equilateral triangles. Since the area of each equilateral triangle is 4^2*sqrt3/ 4 = 4sqrt 3. The area of the hexagon would be 6*4sqrt3 = **24sqrt3.**

6) We can draw altitudes from the vertices of the top base to the feet of the perpendiculars on the longer base. Then, we will have two triangles and a rectangle. The hypotenuse of each of those triangles is 5, while the base is 3. So, the length of the other leg of the triangle is 4. The height of the trapezoid is also 4.

   The area of the trapezoid would be (8+14)/2 * 4 = **44.**

7) 2013 = 3*11*61. There are 8 factors or 4 pairs and each pair multiplies to 2013. So, the product of the divisors would be **2013^4.**

8) 7*9 – (7+9) = 63-16 = **47.**

9) The number of possible outfits is just the product o f 8 *7*3. But, he can choose to wear or not wear his coat. So, there are 2 ways to choose what to do with the coat. Therefore, there are 8*7*3*2 = **336** total combinations.

10) We know that (x+y)(x^2 + y^2) = x^3 + y^3 + xy(x+y).

    (x+y)^2 = x^2 + y^2 + 2xy

    xy = (49-17)/2 = 16.

    (7)(17) = x^3 + y^3 + 16(7).

    x^3 + y^3 = 119 – 112 = **7.**

11) .32444444…….44 = (324-32)/900 = **73/225.**

12) The surface area of a sphere is defined to be 4*pi*r^2. Since the surface area is 36pi, then the radius of the sphere is 3. The diameter of the sphere is 6, which is the long diagonal of the cube. Hence, each side of the cube has length 6/sqrt 3 = 2sqrt3. Now, we know that the volume is just (2sqrt3)^3 = 8(3)sqrt3 = **24sqrt3**

13) There are 9 digits for a possibility for the first digit, but 10 and 10 for the second and third digits respectively. The fourth and fifth digits are already fixed. Our answer is **900.**

14) There are 900 5-digit palindromes. This is a pretty challenging question; don't be intimidated.

Look at each digit. You notice that the ranges for the units digit and the ten-thousands digit are both 1-9. So, each digit appears 100 times in those two digits. For the thousands and tens digit, the possibilities are from 0-9; so, each digit appears 90 times and same with the hundreds digit. The sum of the digits in the units digit/ten-thousands "" is just 100(1+2+…+9) = 4500.Tens digit/Hundreds digit / Thousands digit: 90(1+2+3+…+9) = 4050. So, we can write the sum as this form:

4500 + (4050*10) + (4050*10^2) + (4050*10^3) + (4500*10^4) = **49,500,000.**

15) Using the fact that d = rt, we can set up two equations:

D = 30(r-c) and D = 20(r+c). Solving for these equations, r = 5c. So, r/c = **5.**

16) This problem reduces to determining the sum of the squares of the first 12 integers. That is just 1^2 + 2^2 + 3^2 + …+ 12^2 = (12)(13)(25)/6 = 650.

17) The prime factorization of 432 is 2^4*3^3.

We can let the first digit be 9. The second digit can be a 8. The next two can be 6 and 1.

Our max 4-digit number is **9861.**

18) The chain allows Eli to travel in a 6 foot circle. The area of the hexagon is 4sqrt3*6 = 24sqrt3. Therefore, the area of the region is **36pi – 24sqrt3.**

19) The region is a rhombus.

First equate abs(3x-12) = 12. Say that y =7. Then, x = 0 or x = 8. So, our points are (0,7) and (8,7) on one case. Then, equate abs(6y-42) = 12. Say that x = 4. Then y =9 or y = 5. Now, we can graph the boundary points of this graph: (0,7), (8,7), (4,5), (4,9). The area of the rhombus would be (8)(4)/2 = **16.**

20) X^81 + x^32 + x^6 + 3x^2 + 1 = q(x)(x+1)(x-1) + r(x).

r(x) = ax + b. When x = 1, 7 = a+b. x = -1, 5 = -a+b. 2b = 12 -→ b = 6, a = 1. **R(x) = x+6.**

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Solutions:

8th Grade Team Round

1) Use the formula 30h – 5.5 m. The degree measure is 30(3) – 5.5(30) = **75.**
2) (53+47)(53-47) + (96+94)(96-94) + (22+18)(22-18) = 100*6 + 190*2 + 40*4 = **1140.**
3) AC = 65. You know that the angles opposite AC in both triangles are right. So, AC is a diameter. Now, we know that the radius is **65/2.**
4) You can either have BGBGBG or GBGBGBGB. There are 2 options here. There are 3! = 6 ways to arrange the girls and 3! = 6 ways to arrange the boys. Our answer is 2*6*6 = **72.**
5) A = rs. Your area is 84 using Heron's formula. The semi perimeter is (13+14+15)/2 = 21. Therefore, the radius is **4.**
6) Just 9 C 4 = 9*8*7*6/24 = **126.** This is because there are 9 options for each digit. You know that the non-leading digits cannot be 0. So, we're talking about arranging 9 digits. There is only 1 way to arrange them such that the resulting 4-digit number is increasing. So, 126 is your answer.
7) The number of days it takes is inversely proportional to the number of farmers. So, 4*4 = 2*x. x = 8. It takes **8** days for the 2 farmers to complete the job.
8) This can get kind of messy because of the number of rotations that there can be. So, in order to account for this, make one of the delegates that has the restrictions sit in an arbitrary seat. You have 2 seats for the other delegate such that this delegate does not sit next to the already seated delegate. Now, you have 3 more people which you can seat in 6 ways. So, your answer is just 2*6 = **12.**
9) Similar triangles. Triangle ECM is similar to triangle EDA. Therefore, 2/6 = MC/4. Hence, MC = 4/3. So, BM = 4- (4/3) = **8/3.**
10) Count them! No easy way to do this one! Your answer is **25.**
11) First of all, you know that there are 900 palindromes. Look at each digit. You notice that the ranges for the units digit and the ten-thousands digit are both 1-9. So, each digit appears 100 times in those two digits. For the thousands and tens digit, the possibilities are from 0-9; so, each digit appears 90 times and same with the hundreds digit.
The sum of the digits in the units digit/ten-thousands "" is just 100(1+2+…+9) = 4500.
Tens digit/Hundreds digit / Thousands digit: 90(1+2+3+…+9) = 4050.
So, we can write the sum as this form:
4500 + (4050*10) + (4050*10^2) + (4050*10^3) + (4500*10^4) = **49,500,000.**
12) The units digit only matters for the numbers 1 – 4. When 5! Comes along, you have a 0 and every other factorial after 5! Has a 0 as its units digit. Therefore, our solution is 1+2+4+6 = 13. So, the units digit is **3.**
13) All we care about is green pepper plants and non-green pepper plants. So, we have 7 non-plants and 5 plants. Say you put the 7 non-plants to start with. There are 8 spots that you can put these 5 green pepper plants (between the non-plants) such that no two-green pepper plants are next to each other. So, 8 C 5 = 8 C 3 = 56. There are 12 C 5 ways to choose the entire row. So, our answer is 56 / (12 C 5) = 56/992 = **7/99.**

14) Notice that these numbers in base 4 are 1, 10, 11, 100,…… Therefore, to get the 50th base 4 number, we can determine 50 in base 2. (Look at the terms: 1 is 1(base 2), 2 is 10(base 2), 3 is 11(base 2), etc… 50 in base 2 is just 110010, the 50th term of the sequence is the base 4 number 110010 which is $4^5 + 4^4 + 4^1$ = **1284.**
15) $N^2 = 2^{24} * 3^{16} * 5^{12}$. There are (24+1)*(16+1)*(12+1) = 25*17*13 = 5525 factors. The number of divisors of n^2 that are greater to n is equal to the number of divisors of n^2 that are less than n. So, our answer is just (5525-1)/2 = **2762.**
16) The two lines are orthogonal. Therefore, the angle is **90.**
17) There are ((99-1)/2) + 1= 50 numbers on this list, in which there are 25 pairs. Since each pair values to -2, our answer is 25*(-2) = **-50.**
18) Visualize a 4*4*4 cube. There are 2 cubes on each edge that follow this property. Since there are 12 edges on this cube, our answer is 2*12 = **24.**
19) (5C2)*(6C2) = 10*15 = **150**
20) $X^{81} + x^{32} + x^6 + 3x^2 + 1 = q(x)(x+1)(x-1) + r(x)$.

    r(x) = ax + b.

    When x = 1, 7 = a+b.

    X = -1, 5 = -a+b.

    2b = 12 -→ b = 6, a = 1.

    R(x) = x+6.